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Research Article

Construction of Twenty-Three Points Second Order Rotatable Design in Three Dimensions Using Trigonometric Functions

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Abstract

In this paper, a novel twenty-three-point second-order rotatable design is formulated utilizing trigonometric functions. Design of experiments plays a crucial role in various industries and research fields to investigate the relationship between multiple variables and their effects on a response variable. In particular, second order rotatable designs are widely used due to their ability to efficiently estimate the main, interaction, and curvature effects. Nevertheless, creating designs with a substantial number of points poses difficulties. This study concentrates on developing a second-order rotatable design with twenty-three points utilizing trigonometric functions. Trigonometric functions offer a systematic approach to distribute the points uniformly in the design space, thereby ensuring the optimal coverage of the experimental region. The proposed construction utilizes the properties of sine and cosine functions to generate a balanced and efficient design. The methodology involves dividing the design space into equidistant sectors and assigning the points using the trigonometric functions. By carefully selecting the starting angle and the angular increment, a complete and orthogonal design is achieved. The design is rotatable, meaning it can be rotated to any desired orientation without impairing the statistical properties of the design. Through this construction, the design effectively captures the main effects, interaction effects, and curvature effects. This enables reliable estimation of the model parameters, leading to accurate predictions and efficient optimization. Additionally, the design is efficient in terms of minimizing the number of experimental runs required, thereby reducing costs and time. The suggested second-order rotatable design comprising twenty-three points and employing trigonometric functions exhibits its superiority when compared to conventional designs. It offers a systematic and straightforward approach to construct a balanced and efficient design for studying the relationships between multiple variables. The design's rotatability ensures flexibility in experimental settings, making it a valuable tool for researchers and practitioners in various fields.

Keywords

Curvature Effects, Sine and Cosine Functions, Augmented

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1. Introduction

The construction of designs plays a crucial role in the field of statistical experimental design [11]. It enables researchers to systematically and proficiently assess the impact of different factors on a response variable. According to Marshall et al [13], one popular type of design is the second order rotatable design, which provides a balanced and efficient framework for experimentation. In this article, our attention is directed towards the creation of a twenty-three points second order rotatable design using trigonometric functions. This design is particularly advantageous as it maximizes the efficiency of the experiment while maintaining the balance among the factors. Trigonometric functions, such as sine and cosine, are commonly used in the construction of rotatable designs due to their symmetry properties [10]. By utilizing these functions, we can ensure that the design is rotationally invariant, meaning that the estimates of the treatment effects do not depend on the orientation of the axes.

The development of the second-order rotatable design comprising twenty-three points through trigonometric functions entails identifying the coordinate points positioned on a spherical surface. These points are then transformed to form a design matrix that meets the requirements of balance and rotatability [12]. This process involves carefully selecting the radius of the sphere and the angles of rotation to ensure a uniform spread of points across the design space. Once established, this design facilitates efficient and precise estimation of the primary effects and interactions among the experiment's factors. Additionally, it offers adaptability regarding the number of factors and their respective levels that can be incorporated [2]. The notion of rotatability which is vital in response surface methodology, was first introduced by Box and Hunter [8]. They formulated second-order rotatable designs grounded on geometric arrangements. Bose and Draper [4] underscore the extensive application of response surface fitting methods to assist in the statistical analysis of experimental endeavors, where the product's response is contingent upon one or more controllable variables in an uncertain manner.

This paper introduces a novel three-dimensional second-order rotatable design, consisting of twenty-three points, created using trigonometric functions. The establishment of this second-order rotatable design, which involves twenty-three points and incorporates trigonometric functions, offers a reliable and robust approach to experimental design. Its utilization optimizes experimental resources while still providing valuable insights into the relationship between factors and responses. This design holds great potential for various fields, including manufacturing, pharmaceuticals, and agriculture, where designing efficient experiments is crucial for scientific advancements and decision-making processes [1].

2. Conditions for Second Order Rotatability

We introduce the typical second-order design as;

$$y = \beta_0 + \sum_{i=1}^{k} \beta_i x_i + \sum_{i=1}^{k} \beta_{ii} x_i^2 + \sum\sum_{i\angle j}^{k} \beta_{ij} x_i x_j + \varepsilon \quad (1)$$

Box and Hunter (1957), meeting the following criteria ensures the attainment of a second-order response surface:

$$\sum_{u=1}^{N} x_{iu}^2 = N\lambda_2$$

$$\sum_{u=1}^{N} x_{iu}^4 = 3N\lambda_4 \quad (2)$$

$$\sum_{u=1}^{N} x_{iu}^2 x_{ju}^2 = N\lambda_4$$

$$\sum_{u=1}^{N} x_{iu}^4 = 3\sum_{u=1}^{N} x_{iu}^2 x_{ju}^2, \ (i< j = 1,2,\ldots k)$$

A group of points is deemed to constitute a second-order rotatable design when the aforementioned conditions are met, alongside the non-singularity of the matrix $X'X$ utilized in the least square estimation [7]. Box and Hunter [8] demonstrated that the necessary and adequate condition for this occurrence is,

$$\frac{\lambda_4}{\lambda_2^2} > \frac{k}{k+2} \quad (3)$$

3. The Development of ASORD Through Trigonometric Function Transformations

Bose and Draper [3] were the pioneers in utilizing trigonometric functions for creating second-order rotatable designs. They introduced transformations in the following format:

$$T_1 = \begin{bmatrix} Cos\ \alpha, -Sin \propto, 0 \\ sin\ \propto, cos \propto, 0 \\ 0\ 0\ -1 \end{bmatrix} \quad (4)$$

and

$$T_2 = \begin{bmatrix} Cos\ \frac{\propto}{2}, Sin\frac{\propto}{2}, 0 \\ sin\ \frac{\propto}{2}, cos\frac{\propto}{2}, 0 \\ 0\ 0\ -1 \end{bmatrix} \quad (5)$$

Where $\propto = \frac{2\pi}{s}$

These transformations are applied in the current paper to construct second order rotatable designs.

We designate the construction of 3s – Points as T(r, o, b), with corresponding coordinates.

$$(\mathrm{r}\cos t\alpha \; r\sin t\alpha \; 0)$$
$$(\mathrm{r}\sin t\propto \; 0 \; \mathrm{r}\cos t\propto) \quad (6)$$
$$(0 \; \mathrm{r}\cos t\propto \; \mathrm{r}\sin t\propto)$$

For t = 0, 1, 2… (s-1) and s≥5, When $S \geq 5$ set $T_0(r, 0{,}0)$ and points (rcostα, rsintα, 0) = $3S$

The sums and products of the set up to power four for the co-ordinates listed in (6) are given by;

$$\sum_{u=1}^{N} x_{iu}^2 = sr^2,$$
$$\sum_{u=1}^{N} x_{iu}^4 = \frac{3}{4}sr^4, \quad (7)$$
$$\sum_{u=1}^{N} x_{iu}^2 x_{ju}^2 = \frac{1}{8}sr^4,$$

The excess function for T(r,o,o) is given by;

$$\sum_{u=1}^{N} x_{iu}^4 - 3\sum_{u=1}^{N} x_{iu}^2 x_{ju}^2 = \frac{3}{8}\,\mathrm{sr}^4 \quad (8)$$

In this scenario, the surplus of each individual point fluctuates, necessitating an examination of the cumulative impact across all points. Given that its surplus can be manipulated to be either positive or negative based on the selection of r, it becomes feasible to integrate the T(r, 0, 0) set with sets exhibiting both positive and negative surpluses. Likewise, the coordinates for the four-point set is labeled as G (a, a, a), where this set is halved $\frac{1}{2}\,G(a, a, a)$ and points are $(\pm a, \pm a, \pm a)$.

$$\mathrm{G}\,(\mathrm{a}, \mathrm{a}, \mathrm{a})$$
$$\mathrm{G}\,(-\mathrm{a}, \mathrm{a}, \mathrm{a})$$
$$\mathrm{G}\,(\mathrm{a}, -\mathrm{a}, \mathrm{a}) \quad (9)$$
$$\mathrm{G}\,(\mathrm{a}, \mathrm{a}, -\mathrm{a})$$

The sums and products up to power four for G (a, a, a) is given by,

$$\sum_{u=1}^{N} x_{iu}^2 = 4\mathrm{a}^2$$
$$\sum_{u=1}^{N} x_{iu}^4 = 4\mathrm{a}^4 \quad (10)$$
$$\sum_{u=1}^{N} x_{iu}^2 x_{ju}^2 = 4\mathrm{a}^4$$

The excess for the above set of points is given by

$$\sum_{u=1}^{N} x_{iu}^4 - 3\sum_{u=1}^{N} x_{iu}^2 x_{ju}^2 = -8\mathrm{a}^4 \quad (11)$$

By augmenting specific sets of points in (6) and (9), a designs in three dimensions was obtained. The augmentation was given by $3S$+8. But $S \geq 5$ when S =5, Combining $3s$ given in (6) with two class of eight-point set given in (9) to obtain set denoted Z_1 given as given by $3s + \frac{1}{2}\left[\frac{1}{3}G(a_{1,}a_{1,}a_1)\right] + \frac{1}{2}\left[\frac{1}{3}G(a_{2,}a_{2,}a_2)\right]$, by letting $S = 5$, 23 points were obtained.

4. Construction of Twenty-Three Points ASORD

The moment prerequisites for the collection of twenty-three points to constitute a second-order rotatable configuration are derived by incorporating the points obtained from the 3s set and the 8 set, as outlined in (6) and (9), respectively. The moment conditions specified in (7) were applied to the design point delineated in (6) to ascertain its rotatability. Consequently, these conditions yielded:

$$\sum_{U=1}^{N} X_{iu}^2 = Sr^2 = N\lambda_2$$
$$\sum_{u=1}^{N} x_{iu}^4 = \frac{3}{4}Sr^4 = 3N\lambda_4 \quad (12)$$
$$\sum_{u=1}^{N} x_{iu}^2 x_{ju}^2 = \frac{1}{8}sr^4 = N\lambda_4$$
$$\sum_{u=1}^{N} x_{iu}^4 = 3\sum_{u=1}^{N} x_{iu}^2 x_{ju}^2$$

Solving for the excess in (4.1) gave;

$$\sum_{u=1}^{N} x_{iu}^4 - 3\sum_{u=1}^{N} x_{iu}^2 x_{ju}^2 = \frac{3}{4}Sr^4 - \frac{3}{8}sr^4 = \frac{3}{8}sr^4 \quad (13)$$

The moment conditions given in (10) were used in (9) to confirm rotatability;

$$\sum_{u=1}^{N} x_{iu}^2 = 4a_1^2$$
$$\sum_{u=1}^{N} x_{iu}^4 = 4\,a_1^4 \quad (14)$$
$$\sum_{u=1}^{N} x_{iu}^2 x_{ju}^2 = 4a_1^4$$
$$\sum_{u=1}^{N} x_{iu}^4 = 3\sum_{u=1}^{N} x_{iu}^2 x_{ju}^2$$

Solving for excess in (4.3) gave;

$$\sum_{u=1}^{N} x_{iu}^4 - 3\sum_{u=1}^{N} x_{iu}^2 x_{ju}^2 = 4a_1^4 - 12a_1^4 = -8a_1^4 \quad (15)$$

Again, the moment conditions given in (10) were used on the design point given in (9) to confirm rotatability. Thus, these conditions gave.

$$\sum_{u=1}^{N} x_{iu}^2 = 4a_2^2$$
$$\sum_{u=1}^{N} x_{iu}^4 = 4\,a_2^4 \quad (16)$$

$$\sum_{u=1}^{N} x_{iu}^2 x_{ju}^2 = 4a_2^4$$

$$\sum_{u=1}^{N} x_{iu}^4 = 3\sum_{u=1}^{N} x_{iu}^2\, x_{ju}^2$$

Solving for excess in (4.5) gave;

$$\sum_{u=1}^{N} x_{iu}^4 - 3\sum_{u=1}^{N} x_{iu}^2\, x_{ju}^2 = 4a_2^4 - 12a_2^4 = -8a_2^4 \quad (17)$$

Solving (13), (15) and (17) for excess functions when s=5 gave;

$$\frac{15}{8}r^4 - 8a_1^4 - 8a_2^4 = 0 \quad (18)$$

Let

$$a_1^2 = xr^2 \quad (19)$$

$$a_2^2 = yr^2 \quad (20)$$

Substituting (19) and (20) to (18) gave;

$$(x^2 + y^2) = {}^{15}/_{64} \quad (21)$$

Where x is arbitrary and has positive value say 0.05 using a python software to solve equation (21) to obtain.

$$y = 0.482 \text{ and } x = 0.05 \quad (22)$$

These finally gave;

$$\sum_{u=1}^{23} x_{iu}^2 = sr^2 + 4a_1^2 + 4a_2^2 = 23\lambda_2$$

$$\sum_{u=1}^{23} x_{iu}^4 = \frac{3}{4}sr^4 + 4a_1^4 + 4a_2^4 = 69\lambda_4 \quad (23)$$

$$\sum_{u=1}^{23} x_{iu}^2 = 5r^2 + 0.2r^2 + 1.928r^2 = 23\lambda_2$$

$$\lambda_2 = 0.30991r^2 \quad (24)$$

$$\sum_{u=1}^{23} x_{iu}^4 = 3.75r^4 + 0.01r^2 + 0.93r^4 = 69\lambda_4 \quad (25)$$

$$\lambda_4 = 0.06797r^4 \quad (26)$$

The non-singularity condition was;

$$\frac{\lambda_4}{\lambda_2^2} = \frac{0.06797}{(0.30991)^2} = 0.70768 > \frac{k}{k+2} = \frac{3}{5} = 0.6 \quad (27)$$

therefore

$$\frac{\lambda_4}{\lambda_2^2} > \frac{k}{k+2} = 0.70768 > 0.6$$

These points fulfill the non-singularity requirement for second-order rotatability, thereby meeting the criteria to be classified.

***Table 1**. An overview of the moment conditions.*

Set composition of class	**z1 = 3s**	**Z2=G(a_1,a_1,a_1)**	**Z3=G(a_2,a_2,a_2)**	**z1 +z_2+ z3**
Number of points	For s=5 = 15 points	4 points	4 points	23 points
$\sum_{u=1}^{N} x_{iu}^2$	Sr^2	$4a_1^2$	$4a_2^2$	$sr^2 + 4a_1^2 + 4a_2^2$
$\sum_{u=1}^{N} x_{iu}^4$	$\frac{3}{4}sr^4$	$4a_1^4$	$4a_2^4$	$\frac{3}{4}sr^4 + 4a_1^4 + 4a_2^4$
$\sum_{u=1}^{N} x_{iu}^2 x_{ju}^2$	$\frac{1}{8}sr^4$	$4a_1^4$	$4a_2^4$	$\frac{1}{8}sr^4 + 4a_1^4 + 4a_2^4$
$\sum_{u=1}^{N} x_{iu}^4 - 3\sum_{u=1}^{N} x_{iu}^2 x_{ju}^2$	$\frac{3}{8}sr^4$	$-8a_1^4$	$-8a_2^4$	$\frac{3}{8}sr^4 - 8a_1^4 - 8a_2^4$

5. An Application of ASORD

The objective of the experiment was to assess the impact of three organic fertilizer components on the yield of Ipomea batatas through the utilization of the twenty-three-point second-order rotatable design in three dimensions, The fertilizer components and their respective quantities used were;

1) Poultry manure $x1$= 35 gram per hole,
2) Rabbit manure $x2$= 25 gram per hole,
3) Goat manure $x3$=45 gram per hole.

The focus of the experiment was on the average yield, measured in grams per hole, of *Ipomea batatas*, along with the average number of tubers produced from each hole. The initial letter parameters denote the variability in the quantities of factor application owing to soil fertility fluctuations, leading to numerous expressions of the rotatability criterion. As mentioned by Box and Wilson [6], the natural levels of these mineral elements are denoted as γ_{iu}, with Box and Draper [5] establishing a specific design when λ_2 = 1as part of the scaling down condition.

$$x_{iu} = \frac{\gamma_{iu}-\gamma_i}{s_i}$$

$$\gamma_i = \frac{\sum_{i=1}^{n}\gamma_{iu}}{N}$$

$$s_i = \left[\frac{\sum_{u=1}^{N}(\gamma_{iu}-\gamma_i)^2}{N}\right]^{0.5} \quad (28)$$

$$\gamma_{iu} = x_{iu}s_i + \gamma_i$$

$\sum_{u=1}^{N} x_{iu}^2 = N$ and $\sum_{u=1}^{N} x_{iu} = 0$

The complete second order model to be fitted to yield values is,

$$y = \beta_0 + \sum_{u=1}^{23}\beta_i x_i + \sum_{u=1}^{23}\beta_{ii} x_i^2 + \sum_{i<j}^{23}\beta_{ij}x_i x_j + \varepsilon \quad (29)$$

In the table provided, the treatments for the second-order rotatable design comprising twenty-three points in three dimensions are presented with their corresponding coded and natural levels.

***Table 2.** Coded values and natural levels.*

Coded levels			Natural levels			Weight_yield
-1.384437	0	1.384437	34.5653	25	45.3277	52
1.384437	0	1.384437	35.4347	25	45.3277	87
-1.384437	0	-1.384437	34.5653	25	44.6723	73
1.384437	0	-1.384437	35.4347	25	44.6723	55
-1.384437	1.384437	0	34.5653	25.2014	45	74
1.384437	1.384437	0	35.4347	25.2014	45	58
-1.384437	-1.384437	0	34.5653	24.7986	45	92
1.384437	-1.384437	0	35.4347	24.7986	45	75
0	1.384437	-1.384437	35	25.2014	45.3277	72
0	1.384437	1.384437	35	25.2014	44.6723	78
0	-1.384437	1.384437	35	24.7986	45.3277	89
0	-1.384437	-1.384437	35	24.7986	44.6723	72
-1.384437	0	0	34.5653	25	45	65
1.384437	0	0	35.4347	25	45	56
0	-1.384437	0	35	24.7986	45	56
0	-1.384437	0	35	25.2014	45	77
0	0	1.384437	35	25	45.3277	56
0	0	-1.384437	35	25	44.6723	59
-1.384437	0	0	34.5653	25	45	62
1.384437	0	0	35.4347	25	45	60
0	1.384437	0	35	25.2014	45	60
0	-1.384437	1.384437	35	24.7986	45.3277	80

Coded levels			Natural levels			Weight_yield
0	0	-1.384437	35	25	44.6723	68

The second order parameter estimation

The least square method was employed to calculate the regression coefficients, with X representing the design matrix of sweet potatoes utilizing the formula $(X^I X)^{-I} X^I Y$ where

Table 3. Design Matrix.

X=	[,1]	[,2]	[,3]	[,4]	[,5]	[,6]	[,7]	[,8]	[,9]	[,10]
[1,]	1	-1.384437	0.000000	1.384437	1.916667	0.000000	1.916667	0.000000	-1.916667	0.000000
[2,]	1	1.384437	0.000000	1.384437	1.916667	0.000000	1.916667	0.000000	1.916667	0.000000
[3,]	1	-1.384437	0.000000	-1.384437	1.916667	0.000000	1.916667	0.000000	1.916667	0.000000
[4,]	1	1.384437	0.000000	-1.384437	1.916667	0.000000	1.916667	0.000000	1.916667	0.000000
[5,]	1	-1.384437	1.384437	0.000000	1.916667	1.916667	0.000000	-1.916667	0.000000	0.000000
[6,]	1	1.384437	1.384437	0.000000	1.916667	1.916667	0.000000	1.916667	0.000000	0.000000
[7,]	1	-1.384437	-1.384437	0.000000	1.916667	1.916667	0.000000	1.916667	0.000000	0.000000
[8,]	1	1.384437	-1.384437	0.000000	1.916667	1.916667	0.000000	-1.916667	0.000000	0.000000
[9,]	1	0.000000	1.384437	1.384437	0.000000	1.916667	1.916667	0.000000	0.000000	1.916667
[10,]	1	0.000000	1.384437	-1.384437	0.000000	1.916667	1.916667	0.000000	0.000000	-1.916667
[11,]	1	0.000000	-1.384437	1.384437	0.000000	1.916667	1.916667	0.000000	0.000000	-1.916667
[12,]	1	0.000000	-1.384437	-1.384437	0.000000	1.916667	1.916667	0.000000	0.000000	1.916667
[13,]	1	-1.384437	0.000000	0.000000	1.916667	0.000000	0.000000	0.000000	0.000000	0.000000
[14,]	1	1.384437	0.000000	0.000000	1.916667	0.000000	0.000000	0.000000	0.000000	0.000000
[15,]	1	0.000000	-1.384437	0.000000	0.000000	1.916667	0.000000	0.000000	0.000000	0.000000
[16,]	1	0.000000	1.384437	0.000000	0.000000	1.916667	0.000000	0.000000	0.000000	0.000000
[17,]	1	0.000000	0.000000	1.384437	0.000000	0.000000	1.916667	0.000000	0.000000	0.000000
[18,]	1	0.000000	0.000000	-1.384437	0.000000	0.000000	1.916667	0.000000	0.000000	0.000000
[19,]	1	-1.384437	0.000000	0.000000	1.916667	0.000000	0.000000	0.000000	0.000000	0.000000
[20,]	1	1.384437	0.000000	0.000000	1.916667	0.000000	0.000000	0.000000	0.000000	0.000000
[21,]	1	0.000000	1.384437	0.000000	0.000000	1.916667	0.000000	0.000000	0.000000	0.000000
[22,]	1	0.000000	-1.384437	1.384437	0.000000	1.916667	1.916667	0.000000	0.000000	-1.916667
[23,]	1	0.000000	0.000000	-1.384437	0.000000	0.000000	1.916667	0.000000	0.000000	0.000000

Taking a transpose of X above and multiple it again with elements of matrix X the study obtained

Table 4. Transpose of design Matrix.

M=	[,1]	[,2]	[,3]	[,4]	[,5]	[,6]	[,7]	[,8]	[,9]	[,10]
[1,]	23.000000	0.00000	0.000000	0.000000	23.000000	23.000000	23.000000	0.00000	3.833333	-1.916667

M=	[,1]	[,2]	[,3]	[,4]	[,5]	[,6]	[,7]	[,8]	[,9]	[,10]
[2,]	0.000000	23.00000	0.000000	0.000000	0.000000	0.000000	0.000000	0.00000	5.307010	0.000000
[3,]	0.000000	0.00000	23.000000	-1.916667	0.000000	0.000000	-2.653505	0.00000	0.000000	2.653505
[4,]	0.000000	0.00000	-1.916667	23.000000	0.000000	2.653505	0.000000	0.00000	-5.307010	-2.653505
[5,]	23.000000	0.00000	0.000000	0.000000	44.083333	14.694444	14.694444	0.00000	7.347222	0.000000
[6,]	23.000000	0.00000	0.000000	2.653505	14.694444	44.083333	18.368056	0.00000	0.000000	-3.673611
[7,]	23.000000	0.00000	-2.653505	0.000000	14.694444	18.368056	44.083333	0.00000	7.347222	-3.673611
[8,]	0.000000	0.00000	0.000000	0.000000	0.000000	0.000000	0.000000	14.69444	0.000000	0.000000
[9,]	3.833333	5.30701	0.000000	-5.307010	7.347222	0.000000	7.347222	0.00000	14.694444	0.000000
[10,]	-1.916667	0.00000	2.653505	-2.653505	0.000000	-3.673611	-3.673611	0.00000	0.000000	18.368056

Then the inverse of M was given as:

Table 5. Inverse of Design Matrix.

N=	[,1]	[,2]	[,3]	[,4]	[,5]	[,6]	[,7]	[,8]	[,9]	[,10]
[1,]	0.51440	-0.010416	-0.015942	0.02581	-0.17459	-0.146033	-0.15791	0.00000	0.04514	-0.001080
[2,]	-0.01041	0.048992	0.000377	-0.00560	0.006946	0.000562	0.00685	0.0000	-0.02389	-0.000467
[3,]	-0.01594	0.000377	0.045268	0.00242	0.00501	0.002667	0.008056	0.0000	-0.0016	-0.005707
[4,]	0.025811	-0.00560	0.002428	0.05093	-0.01159	-0.00815	-0.00958	0.0000	0.024277	0.006150
[5,]	-0.17459	0.00694	0.00501	-0.01159	0.08848	0.04149	0.04942	0.0000	-0.03010	-0.00243422
[6,]	-0.14603	0.00056	0.00266	-0.00815	0.04149	0.072026	0.03326	0.0000	-0.00243	0.004256
[7,]	-0.15791	0.00685	0.00805	-0.0095	0.04942	0.03326	0.0804	0.0000	-0.02969	0.003723
[8,]	0.00000	0.00000	0.000000	0.0000	0.0000	0.00000	0.00000	0.06805	0.00000	0.00000
[9,]	0.045145	-0.02389	-0.001636	0.0242	-0.03010	-0.00243	-0.02969	0.0000	0.10357	0.002027
[10,]	-0.00108	-0.00046	-0.00570	0.00615	-0.00243	0.00425	0.0037	0.0000	0.00202	0.0576387

N obtained above was then multiplied with a transpose of matrix X and the list of elements of the vector Y as shown below to obtain;

P= [1576.000000 -37.379807 -62.299679 42.917557 1550.583336 1692.416670 1611.916669 1.916667 312.416667 -197.416667]

The estimation of the regression coefficients was accomplished by multiplying the elements in N with the list P. This process yielded the regression coefficients for the second-order polynomial model as follows:

Table 6. Coded Coefficients.

Term	Coef	SE Coef	T-Value	P-Value	VIF
Constant	52.27	6.23	8.40	0.000	
Poultry	-2.25	2.55	-0.88	0.395	1.00
Rabbit	-2.57	2.61	-0.99	0.341	1.04
Goat	0.73	2.61	0.28	0.783	1.04

Term	Coef	SE Coef	T-Value	P-Value	VIF
Poultry *Poultry	7.34	4.79	1.53	0.149	1.68
Rabbit *Rabbit	14.83	4.55	3.26	0.006	1.52
Goat*Goat	8.57	4.55	1.88	0.082	1.52
Poultry *Rabbit	0.25	4.43	0.06	0.956	1.00
Poultry *Goat	13.25	4.43	2.99	0.010	1.00
Rabbit *Goat	-4.80	4.07	-1.18	0.259	1.05

This section details the practical estimation of regression coefficients within a model designed to accommodate a second-order equation (29). The model focuses on predicting the yield of sweet potato, with poultry, rabbit, and goat manure serving as explanatory variables. The methodology employed for developing the yield model is Response Surface Methodology.

The regression coefficients and their significance levels are presented in table 6. The analysis reveals that the terms involving $x_3$and all quadratic terms are statistically significant at a significance level of (P < 0.05). Additionally, terms involving the interaction between x_1x_2, as well as x_1x_3, remain significant at a significance level of (P < 0.10).

Further regression analysis of the experimental data table 7 indicates that goat manure has a significant positive linear effect on sweet potato yield. Among the three parameters considered, goat manure exhibits the highest positive linear effect (0.73, p=0.783), followed by poultry, while rabbit manure shows a negative linear effect.

Interactions between poultry and rabbit, as well as poultry and goat, demonstrate a strong positive effect on sweet potato yield, suggesting that yield increases with higher levels of these factors. Conversely, the interaction between rabbit and goat shows a negative impact, indicating a decrease in yield with increasing levels of these factors.

The study also observes that quadratic terms involving all factors are significant, which aligns with findings from a previous study conducted by [9]. In his research, he employed a Randomized Complete Block Design, conducting three replications, and included treatments comprising both cattle manure and poultry manure.

From table 6, the fitted model therefore was given as;

$$\hat{y} = 52.27 - 2.25x_1 - 2.57x_2 + 0.73x_3 + 7.34x_1^2 + 14.83x_2^2 + 8.37x_3^2 + 0.25x_1x_2 + 13.25x_1x_3 - 4.80x_2x_3$$

Table 7. *Analysis of variance for sweet potatoes yield.*

Source	DF	Adj SS	Adj MS	F-Value	P-Value
Model	9	2031.41	225.712	2.88	0.041
Linear	3	146.42	48.806	0.62	0.613
Poultry	1	60.75	60.750	0.78	0.395
Rabbit	1	76.44	76.445	0.98	0.341
Goat	1	6.19	6.189	0.08	0.783
Square	3	866.40	288.799	3.69	0.040
Poultry *Poultry	1	184.09	184.094	2.35	0.149
Rabbit *Rabbit	1	831.89	831.888	10.62	0.006
Goat *Goat	1	277.61	277.613	3.54	0.082
2-Way Interaction	3	811.58	270.526	3.45	0.048
Poultry *Rabbit	1	0.25	0.250	0.00	0.956
Poultry*Goat	1	702.25	702.250	8.96	0.010
Rabbit *Goat	1	109.08	109.078	1.39	0.259

Source	DF	Adj SS	Adj MS	F-Value	P-Value
Error	13	1018.33	78.333		
Lack-of-Fit	8	780.33	97.541	2.05	0.223
Pure Error	5	238.00	47.600		
Total	22	3049.74			

The analysis of variance indicates that there are significant using the *p*-values for linear terms also point out that their contribution is significant to the model.

Table 8. *Model summary.*

Model Summary		
S	R.sq	R.sq (adj)
8.85061	86.61%	73.49%

The results for the adjusted R^2 indicate that 73.49% (0.7349) of the variation in the response was explained by the model.

As the response surface can be described by a second-order model, it was important to examine the optimal conditions. Graphical representation is highly beneficial for grasping the behavior of the second-order response surface [15]. In particular, contour plots aid in identifying the surface's shape and approximating the optimal response. The contour plot depicting the yield of sweet potatoes was presented in figure 1 and the surface plot in figure 2.

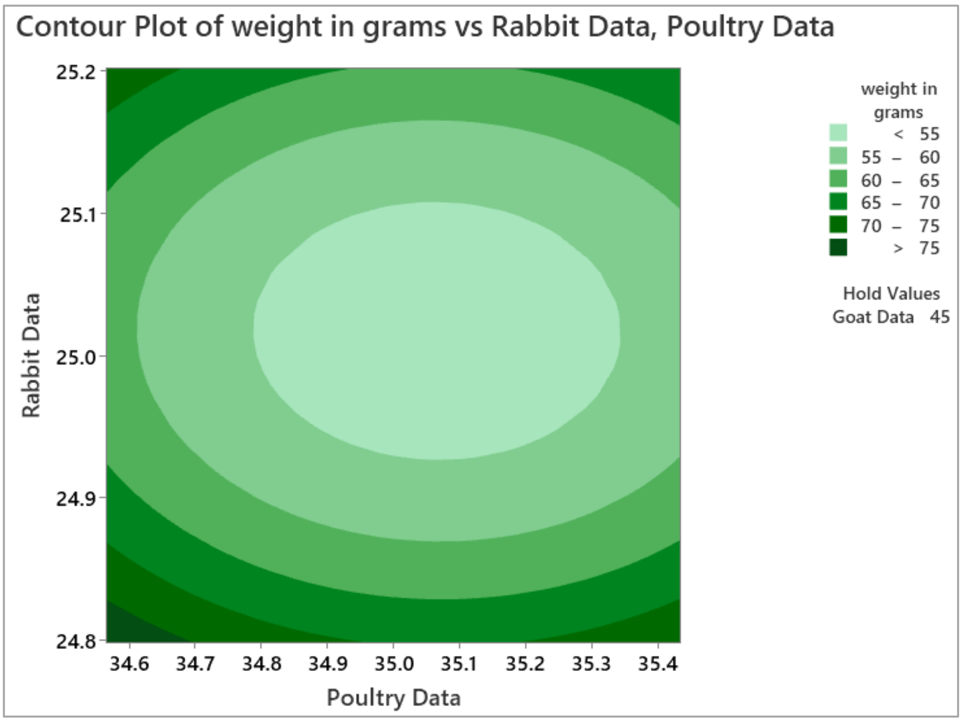

Figure 1. *Contour plot.*

The contour plot of weight in grams vs. rabbit data and poultry data used for the growth of sweet potatoes, illustrates how the weight of the sweet potatoes varies based on the combination of factors such as rabbit manure and poultry manure. Each contour line on the plot represents a constant weight value of the sweet potatoes. By analyzing this plot, we can observe how different combinations of rabbit and poultry manure impact the weight of the sweet potatoes. This visualization helps in understanding the relationship between the types of manure used and the resulting growth of sweet potatoes, aiding in optimizing agricultural practices for better yields [14].

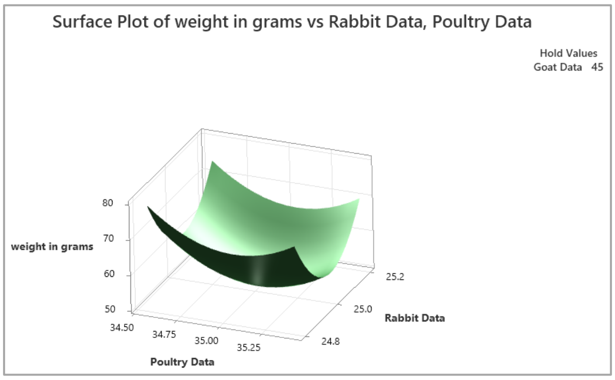

Figure 2. *Surface plot.*

A surface plot depicting the effect of poultry and rabbit manure on the growth of sweet potatoes illustrates how the combination of these manures influences the yield of sweet potatoes. The plot visualizes how varying proportions of poultry and rabbit manure impact the growth of sweet potatoes, with each point on the surface representing a specific combination of the two types of manure and the resulting yield of sweet potatoes. This visualization helps in understanding the relationship between the composition of manure and the productivity of sweet potato crops, enabling farmers to optimize their use of manure for better yields.

6. Conclusions

Based on the analysis of the given second-order model, it can be concluded that all three types of effects, namely linear, pure quadratic, and interaction, exhibit significance in predicting the average weight yield of sweet potatoes. Specifically, the main effects of poultry manure (x_1) and rabbit manure (x_2), along with their quadratic terms (x_1^2 and x_2^2), demonstrate substantial influence on the predicted yield, as evidenced by their relatively large coefficients. The main effect of goat manure (x_3) appears to have a smaller impact on the yield, as indicated by its lower coefficient magnitude.

Furthermore, significant pure quadratic effects suggest that the relationship between the independent variables and the yield is not purely linear but exhibits curvature. This is particularly evident in the cases of x_1^2 and x_2^2 where their coefficients imply pronounced quadratic effects.

While the current study provides valuable insights into the effects of different types of manure on sweet potato yield, further research and experimentation may be warranted. Investigating additional factors or variables that could influence yield, such as soil composition, climate conditions, or alternative fertilization methods, can contribute to a more comprehensive understanding of yield determinants and aid in refining agricultural practices for sweet potato cultivation.

Abbreviations

ASORD	Augmented Second Order Rotatable Design

Author Contributions

Dennis Mwan Matundura: Conceptualization, Software, Formal analysis, Writing-original draft, Methodology, Visualization

Mathew Kosgei: Conceptualization, Supervision, Methodology, Writing-review and editing

Robert Too: Methodology, Project administration, Supervision, Writing – review & editing

Conflicts of Interest

The authors declare no conflicts of interest.

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