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# Bayesian Interval Estimation in a Non-Homogeneous Poisson Process with Delayed S-Shaped Intensity Function

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**Abstract:** Software reliability assessment has been explored by many researchers over the past decades. With the increasing development of new complex software systems, accurate methods for estimating reliability model parameters are needed. Facilitated by the increasing use of computer systems in various sectors such as air traffic control, banking, industrial processes, and government operations, developing accurate reliability assessment methods is indispensable. The Delayed S-shaped software reliability model is one of the non-homogeneous Poisson process (NHPP) software reliability models proposed for capturing error detection and removal processes in software reliability testing. Many researchers have fitted the model to software failure data and performed estimation using the Maximum Likelihood method and Bayesian approach, however, construction of Bayesian credible sets for the parameters of this model and comparison of their efficiencies with the Wald confidence intervals using simulation have not been explored. The Bayesian interval estimation was conducted with three different joint prior distributions assigned to the parameters $\alpha$ and $\beta$ of the model, namely the gamma distributed informative prior and, $1/\alpha$, and $1/\alpha\beta$ as non-informative priors. The Bayesian credible intervals and Wald confidence intervals for the two parameters were compared on the basis of interval lengths and coverage probabilities. The simulation was assumed to emulate the end-user environment and can generate inter-failure times data for the study. The Delayed S-shaped reliability model variables were simulated with fixed parameters set at $(\alpha, \beta)$=(20, 0.5). The hyperparameters for the informative prior were chosen such that they have minimal effect on the results. In other words, the prior information does not swamp the information from the data. The Bayesian method yields superior results, as evidenced by shorter interval lengths and higher coverage probabilities in Table 1.

**Keywords:** Non-Homogeneous Poisson Process, Intensity Function, Software Reliability Model, Informative Priors, Bayesian Method, Wald Intervals, Maximum Likelihood

## 1. Introduction

Computers have increasingly been used in various sectors of contemporary society. Today, complex software systems are evidently and dominantly used in the healthcare sector, engineering, filmmaking industries, architects, education sector, military systems, traffic control in large urban centers, air traffic control, banking, industrial processes, financial sectors, and government operations, among others. The increased dependency has led to rapid changes in software systems, especially the development of more complex systems, posing major challenges to their reliability. Software is reliable if it can accomplish its defined functions under a given environment in a specified time period. Software reliability is the probability of failure-free software operations in a specified time period in a given environment [1]. Software defects can cause system failure, which is avoidable by developing reliable software. Usually, software defects are removed by running tests in a way that emulates the end-user environment, which is costly, challenging, and time-consuming [2]. Thus, software reliability modeling addresses these challenges by providing a framework for assessing current reliability, monitoring reliability assessment progress, and predicting future failures. Software testing is done during the development stage to achieve desired reliability. Software reliability modeling is done by first statistically estimating the parameters of the model selected. Accurate estimation of the parameters is critical, especially because the model, with its estimated parameters, can be used to predict when the next error will occur and when to terminate the development process. The more accurate estimates of the parameters of the model selected, the more likelihood of accurate prediction; hence, reliable software.

Several authors and researchers have developed various software reliability models over the past decades. The Non-homogeneous Poisson Process (NHPP) Delayed S-shaped software reliability model was developed by Yamada, S. etc. [3] and has an intensity function given by;

$$\lambda(t)=\alpha\beta^2 te^{-\beta t} \quad (1)$$

The intensity function represents the error content represented per unit testing time. The model was developed by modifying the Goel-Okumoto (G-O) model to make it S-shaped [4]. The model is appropriate because it emulates the real testing scenario. Based on experience, the cumulative number of errors curve is usually S-shaped [3]. The S-shape occurs because, at the beginning of the testing process, some errors might be covered by others. As such, removing a detected error at the beginning of the test reduces the failure intensity slightly because other errors in the same test data will still cause a failure. Before the detected errors are removed, the ones initially covered remain undetected [5]. The Delayed S-shaped NHPP model also captures the learning process through which software users become well acquainted with the software and test tools. The model assumes that the probability of failure detection at any time is proportional to the current number of faults in the software, the initial error content of the software is a random variable, and detected errors are simultaneously removed without introducing other errors [5]. Further, there is a delay between the time the error is detected and when it is reported [6]. Lai, R. & Garg, M. [7] also argued that the Delayed S-shaped model was developed to account for the delay (lag) in error detection and its removal.

Several researchers have used the Delayed S-shaped software reliability model in the software reliability test. Lee, T. Q. etc. [8] used the Delayed S-shaped software reliability model to propose a Bayesian model for estimating expected test cost and reliability. The researchers extended the Delayed S-shaped model to cope with the situation of insufficient historical data for enhancing software reliability prediction. Even though the researchers used gamma-distributed informative prior, the primary purpose of the research was to estimate cost and reliability using the extended model. Yin, L. & Trivedi Kishor, S. [6] carried out a confidence interval estimation of NHPP-based software reliability models, including the Goel-Okumoto and the Delayed S-shaped NHPP models. The researchers computed Wald confidence intervals and Bayesian methods but used a non-informative prior given by 1/α. The Bayesian approach is advantageous since it allows the combination of prior information with more recent information obtained from field data or tests [9].

This paper focuses on Bayesian and Wald interval estimation for the parameters of the Delayed S-shaped model. The joint gamma-distributed informative prior, 1/αβ, and 1/α were used as prior distributions for Bayesian analysis. Wald confidence intervals for the parameters of the model were first constructed, followed by Bayesian credible intervals. The confidence and credible intervals were compared using interval lengths and coverage probabilities on the basis of simulation.

# 2. Non-Homogeneous Poisson Process with Delayed S-Shaped Intensity Function

Let $N(t)$ denote the number of events occurring in a time interval $(0,t]$. A counting process $\{N(t), t \geq 0\}$ is said to be a non-homogeneous Poisson process (NHPP) with intensity function $\lambda(t)$ if it satisfies the following conditions;

i $N(0) = 0$

ii The process has independent increment property such that $N(t_0,t_1), N(t_1,t_2), \dots, N(t_{n-1},t_n)$ are independent random variables for any specified time points $t_0 < t_1 < t_2 < \cdots < t_n$ where $t_0 = 0$.

iii The probability that n failures occur at a time interval of $(0,t)$ is given by;

$\Pr\{N(t) = n\} = \frac{[m(t)]^n}{n!} e^{-m(t)}$, where m (t) is the mean value function denoting the expected number of failures and is given by; $m(t) = \int_0^t \lambda(t)dt$, and $\lambda(t)$ is the intensity function.

iv The process' failure rate is given by the probability that exactly one failure occurs within a small-time interval $\Delta t$ denoted by

$\Pr(t, t+\Delta t) = \Pr\{N(t, t+\Delta t) - N(t) = 1\} = \lambda(t)\Delta t + o(\Delta t)$, where $\lambda(t)$ is the intensity function.

v The probability that more than one failure occurs within a small interval of time, $\Delta t$, is negligible. i.e., $\Pr\{N(t+\Delta t) - N(t) \geq 2\} = o(\Delta t)$.

The NHPP with Delayed S-shaped intensity function is a software reliability growth model which is commonly used in capturing the removal of software errors. Let $0 < t_1 < t_2 < \cdots < t_n \leq t$ be the software failure times in the time interval $(0,t]$.

# 3. Wald Confidence Intervals

Suppose that $t_1, t_2, \dots, t_n$ denote the $n$ observed failure times in the interval $(0,t]$. Then the likelihood function for the Delayed S-shaped software reliability model with intensity function in (1) is given by;

$$L(\alpha,\beta|\underline{t}) = e^{-m(t)} \prod_{i=1}^{n} \lambda(t_i) = e^{-\alpha(1-(1+\beta T)\exp(-\beta T)} \prod_{i=1}^{n} \left(\alpha\beta^2 t_i e^{-\beta t_i}\right)$$

$$= \alpha^n \beta^{2n} (\prod_{i=1}^{n} t_i) e^{-\beta \sum t_i} e^{-\alpha(1-(1+\beta T)\exp(-\beta T))} \quad (2)$$

The Log-likelihood function of the model is given by;

$$l(\alpha, \beta|\underline{t}) = nlog\alpha + 2nlog\beta + \log(\prod_{i=1}^{n} t_i) - \beta \sum t_i - \alpha(1 - (1 + \beta T)\exp(-\beta T)) \quad (3)$$

The procedure for constructing Wald confidence intervals is described as follows: Differentiate the log-likelihood function $l(\alpha, \beta/\underline{t})$ partially with respect to $\alpha$ and $\beta$ and equate the resultant derivative to zero. This yields the following two equations.

$$\frac{n}{\alpha} - \{1 - (1 + \beta T)e^{-\beta T}\} = 0 \quad (4)$$

and;

$$\frac{2n}{\beta} - (\sum_{i=1}^{n} t_i) - \alpha\beta T^2 e^{-\beta T} = 0 \quad (5)$$

Equations (4) and (5) are then simultaneously solved numerically to obtain the maximum likelihood estimates $\hat{\alpha}$ and $\hat{\beta}$ of the parameters $\alpha$ and $\beta$.

The asymptotic variances of the estimates of $\alpha$ and $\beta$ are obtained from the inverse observed Fisher Information matrix, $I(\hat{\alpha}, \hat{\beta})$, which is the matrix containing negative second order derivatives of the log-likelihood function in (3).

$$I(\hat{\alpha}, \hat{\beta}) = \begin{bmatrix} \frac{\partial^2 logL(\alpha,\beta|\underline{t})}{\partial \alpha^2} & \frac{\partial^2 logL(\alpha,\beta|\underline{t})}{\partial \alpha \partial \beta} \\ \frac{\partial^2 logL(\alpha,\beta|\underline{t})}{\partial \alpha \partial \beta} & \frac{\partial^2 logL(\alpha,\beta|\underline{t})}{\partial \beta^2} \end{bmatrix}_{(\alpha,\beta)=(\hat{\alpha},\hat{\beta})} \quad (6)$$

where

$$\frac{\partial^2 logL(\alpha,\beta|\underline{t})}{\partial \alpha^2} = -\frac{n}{\alpha^2} \quad (7)$$

$$\frac{\partial^2 logL(\alpha,\beta|\underline{t})}{\partial \beta^2} = -\frac{2n}{\beta^2} - \alpha(1 + \beta T)T^2 e^{-\beta T} \quad (8)$$

$$\frac{\partial^2 logf(\alpha,\beta|\underline{t})}{\partial \alpha \partial \beta} = -\beta T^2 e^{-\beta T} \quad (9)$$

Suppose that the inverse of the observed Fisher information $I(\hat{\alpha}, \hat{\beta})$ in (6) is given as

$$I(\hat{\alpha}, \hat{\beta})^{-1} = \begin{bmatrix} \hat{\sigma}_{11}(\hat{\alpha}, \hat{\beta}) & \hat{\sigma}_{12}(\hat{\alpha}, \hat{\beta}) \\ \hat{\sigma}_{21}(\hat{\alpha}, \hat{\beta}) & \hat{\sigma}_{22}(\hat{\alpha}, \hat{\beta}) \end{bmatrix} \quad (10)$$

Therefore, the $100(1 - \alpha)\%$ Wald confidence interval for $\alpha$ can be constructed as

$$\hat{\alpha} \pm Z_{\frac{\alpha}{2}}[\hat{\sigma}_{11}(\hat{\alpha}, \hat{\beta})]^{\frac{1}{2}} \quad (11)$$

Similarly, the $100(1 - \alpha)\%$ Wald confidence interval for $\alpha$ can be constructed as

$$\hat{\beta} \pm Z_{\frac{\alpha}{2}}[\hat{\sigma}_{22}(\hat{\alpha}, \hat{\beta})]^{\frac{1}{2}} \quad (12)$$

# 4. Bayesian Credible Intervals

The Bayesian method has been adopted in the parameter estimation of software reliability models. When testing software reliability, researchers may be unable to obtain sufficient historical data, leading to small sample sizes, which are best used for estimating software reliability parameters by using the Bayesian approach [8]. The method depends on prior information, enabling researchers to incorporate expert opinions from previous studies into the current study [11]. The Bayes rule outlines that the accurate posterior estimate is obtained by combining prior knowledge with noisy sensory inputs (often represented by the likelihood function) as presented in the equation below [11]:

$$P(\underline{\theta}/\underline{x}) = \pi(\underline{\theta}) * L(\underline{\theta}/\underline{x}) \quad (13)$$

The approach implies that the first step entails defining a probability distribution for the data, and the second involves choosing appropriate prior distributions for the model's parameters. The choice of the priors is often guided by the available information about the parameters of interest or intuition [10]. Although informative priors have been commonly used, theory has outlined methods for obtaining non-informative priors to address the limitation of being guided by intuitive knowledge. Such methods include using flat, conjugate, reference, and Jeffrey's priors. In this paper, three different priors were used: a gamma-distributed informative prior, a non-informative prior given by $1/\alpha$, used by Yin and Trivedi [6] with the same model, and a non-informative prior given by $1/\alpha\beta$.

Although the Bayesian method has proven to be efficient in estimation, the posterior distribution may often be intractable such that it is impossible to perform exact sampling from it [12]. Moreover, the high dimensionality of the posterior distribution may require focusing on the marginal posterior distribution of each parameter, obtained by integrating out over the other parameters in the model [13]. One way to circumvent this problem is to use the Gibbs sampling technique by first obtaining full conditional distributions for each parameter [14]. However, if at least one of the full conditional distributions is of unknown distribution (difficult to sample from directly), samples can be generated using a Metropolis-Hastings-type Markov Chain Monte Carlo (MCMC) method. The Metropolis-Hastings MCMC approach consists of two steps [12]: (i) generate a new sample given the previous sample using some proposal distribution; (ii) compare the likelihood of the new sample to that of the previous sample to accept (use it for inference) or reject (use the previous sample again) the proposed sample.

### *4.1. Gamma-Distributed Informative Prior*

We adopted the joint gamma-distributed prior for $\alpha$ and $\beta$, where the parameters are distributed as $Gamma(a, b)$ and $Gamma(c, d)$ , respectively and are assumed to be independent. The joint prior is then given by

$$\pi(\alpha,\beta) \propto \alpha^{a-1}\beta^{c-1}e^{-(b\alpha+d\beta)}, \alpha > 0, \beta > 0 \quad (14)$$

where a, b, c, and d, are hyper-parameters.

Given the vector of observed software failure times $\underline{t}$, the corresponding posterior distribution for α and β was obtained using (13) as follows;

$$\pi(\alpha,\beta|\underline{t}) \propto \pi(\alpha,\beta)L(\alpha,\beta|\underline{t}) \propto \alpha^{a-1}\beta^{c-1}e^{-(b\alpha+d\beta)}\alpha^{n}\beta^{2n}(\prod_{i=1}^{n} t_i)e^{-\beta\sum t_i}e^{-\alpha(1-(1+\beta T)\exp(-\beta T)}$$

$$\propto \alpha^{n+a-1}\beta^{2n+c-1}e^{-\alpha[b+1-(1+\beta T)e^{-\beta T}]-\beta[d+\sum_{i=1}^{n} t_i]} \quad (15)$$

The full conditional distributions were derived from the joint posterior distribution using the following formulas:

$$p(\alpha|\beta,\underline{t}) = \frac{P(\alpha,\beta|\underline{t})}{\int P(\alpha,\beta|\underline{t})d\alpha} \quad (16)$$

$$p(\beta|\alpha,\underline{t}) = \frac{P(\alpha,\beta|\underline{t})}{\int P(\alpha,\beta|\underline{t})d\beta} \quad (17)$$

Using (16), the full conditional for alpha was obtained as;

$$p(\alpha|\beta,\underline{t}) \propto \alpha^{n+a-1}e^{-\alpha(b+1-(1+\beta T)e^{-\beta T})} \quad (18)$$

which is a kernel of gamma distribution with shape parameter $(n+a)$ and scale parameter $b+1-(1+\beta T)e^{-\beta T}$.

Similarly, using (17), the full conditional for $\beta$ was obtained as;

$$\pi(\beta|\alpha,\underline{t}) \propto \beta^{2n+c-1}e^{\alpha(1+\beta T)e^{-\beta T}}e^{-\beta[d+\sum_{i=1}^{n} t_i]} \quad (19)$$

which is of unknown distribution.

### *4.2. Prior Function* $1/\alpha\beta$

Using this prior and (2) and (13), the posterior distribution was obtained as follows;

$$\pi(\alpha,\beta|\underline{t}) \propto \frac{1}{\alpha\beta}\alpha^{n}\beta^{2n}(\prod_{i=1}^{n} t_i)e^{-\beta\sum t_i}e^{-\alpha(1-(1+\beta T)\exp(-\beta T))}$$

$$\propto \alpha^{n-1}\beta^{2n-1}e^{-\alpha[1-(1+\beta T)e^{-\beta T}]-\beta\sum_{i=1}^{n} t_i} \quad (20)$$

The full conditional distributions were obtained using (16) and (17) as follows;

$$p(\alpha|\beta,\underline{t}) \propto \alpha^{n-1}e^{-\alpha(1-(1+\beta T)e^{-\beta T})} \quad (21)$$

which is a kernel of gamma distribution with shape parameter, n, and scale parameter $(1-(1+\beta T)e^{-\beta T})$.

$$\pi(\beta|\alpha,\underline{t}) \propto \beta^{2n-1}e^{\alpha(1+\beta T)e^{-\beta T}}e^{-\beta\sum_{i=1}^{n} t_i} \quad (22)$$

which is of unknown distribution.

### *4.3. Prior Function* $1/\alpha$

For this prior, the posterior density function was obtained using (2) and (13) as follows;

$$\pi(\alpha,\beta|\underline{t}) \propto \frac{1}{\alpha}.\alpha^{n}\beta^{2n}(\prod_{i=1}^{n} t_i)e^{-\beta\sum t_i}e^{-\alpha(1-(1+\beta T)\exp(-\beta T))} \propto \alpha^{n-1}\beta^{2n}e^{-\alpha[1-(1+\beta T)e^{-\beta T}]-\beta\sum_{i=1}^{n} t_i} \quad (23)$$

The following full conditional distributions were obtained from the posterior density function using (16) and (17).

$$\pi(\alpha|\beta,\underline{t}) \propto \alpha^{n-1}e^{-\alpha(1-(1+\beta T)e^{-\beta T})} \quad (24)$$

which is a kernel of gamma distribution with shape parameter, n, and scale parameter $(1-(1+\beta T)e^{-\beta T})$.

$$\pi(\beta|\alpha,\underline{t}) \propto \beta^{2n}e^{\alpha(1+\beta T)e^{-\beta T}}e^{-\beta\sum_{i=1}^{n} t_i} \quad (25)$$

which is of unknown distribution.

We note that it is difficult to sample directly from the derived joint posterior distributions in (15), (20), and (13). A possible way of simulating these posterior densities is by using their corresponding full conditional distributions for the two parameters α given β, using the Gibbs sampling approach. However, the full conditional distributions $\pi(\beta/\alpha,\underline{t})$ for $\beta$ are improper for all the three types of priors considered, showing that it may be difficult to sample from them directly, necessitating the use of the Metropolis-Hasting MCMC approach. In this regard, the Metropolis-Hasting step within the Gibbs sampling technique was used to sample from the posterior distributions using the following algorithm.

*Step* 1: Start with $k=1$ and set the initial values of

$\{\underline{\theta}^{(1)} = (\alpha^{(1)}, \beta^{(1)})\}$

*Step* 2: Sample $\alpha_1$ from $\alpha_k \sim p(\alpha|\beta_{k-1}, \underline{t})$ (which is gamma-distributed) to have the current state as $(\alpha_1, \beta_0)$.

*Step* 3: Using the proposal distribution of $\beta$, where the proposal was chosen as $\beta \sim N(\beta^{(k-1)}, \sigma_\beta^2)$, sample a candidate value for $\beta^{prop}$ using $\alpha_1$ obtained in step two.

*Step* 4: Generate $u \sim UNIF(0,1)$

*Step* 5: Compute the MH acceptance ratio at the candidate value, $\underline{\theta}^{prop}$, and the previous value, $\underline{\theta}^{(k-1)}$, using block updating.

$$R_{\underline{\theta}} = \frac{\pi_{\underline{\theta}}(\underline{\theta}^{prop}|\,\underline{t}) * q_{\underline{\theta}}(\underline{\theta}^{(k-1)}|\,\underline{\theta}^{prop})}{\pi_{\underline{\theta}}(\underline{\theta}^{(k-1)}|\,\underline{t}) * q_{\underline{\theta}}(\underline{\theta}^{prop}|\,\underline{\theta}^{(k-1)})}$$

*Step* 6: If $u \leq \min(1, R_{\underline{\theta}})$, accept the candidate point $(\beta_k)$ with probability $\min(1, R_{\underline{\theta}})$: set $\underline{\theta}^{(k)} = \underline{\theta}^{prop}$. Otherwise set $\underline{\theta}^{(k)} = \underline{\theta}^{(k-1)}$ to have the current state $(\alpha_1, \beta_1)$.

*Step* 7: Repeat steps 2 to 6 until the desired sample size, k, is obtained.

On the basis of the simulation of the joint posterior distribution of $\alpha$ and $\beta$ histograms are constructed and $100(1-\alpha)\%$ credible intervals for the two parameters are computed as follows: Let $\gamma \in (0,1)$. If $\beta_L^*$ and $\beta_U^*$ are respectively the $\frac{\gamma}{2}$ and $\left(1-\frac{\gamma}{2}\right)$ posterior quantiles for $\beta$, then $(\beta_L^*, \beta_U^*)$ is a $100(1-\gamma)\%$ credible interval for $\beta$. Similarly, if $\alpha_L^*$ and $\alpha_U^*$ are respectively the $\frac{\gamma}{2}$ and $\left(1-\frac{\gamma}{2}\right)$ posterior quantiles for $\alpha$, then $(\alpha_L^*, \alpha_U^*)$ is a $100(1-\gamma)\%$ credible interval for $\alpha$.

## 5. Simulation Study

The study investigated the performance of the Wald and Bayesian interval estimation methods on the basis of data simulated from the Delayed S-shaped NHPP model's intensity function using the thinning algorithm provided by Lewis, P. W. & Shedler, G. S. [15]. It was assumed that the simulation process emulates the end-user environment and can generate inter-failure times data for reliability testing. An R code was developed to simulate interfailure times data when T = 100, $\lambda_0$ = 0.4, $\alpha_0$ = 20, and $\beta_0$ = 0.05, where $\lambda_0$ is the error content rate at time t = 0. The simulated data was used to sample α and β random variables from the posterior distributions using the Metropolis Hasting MCMC procedure as explained above. For the Bayesian method with informative prior, the hyperparameters were chosen such that they have minimal effect on the results. In other words, the prior information does not swamp the information from the data. Furthermore, since Metropolis Hasting MCMC was used, we proposed a normal distribution for β and set the unknown parameter theta at θ = 0.05. Once the data for α and β were obtained, they were assessed to determine general patterns. First, trace plots were constructed for α and β random variables to examine stability. As shown in Figure 1, the blue trace plots were generated using the gamma-distributed informative prior, the red trace plots were constructed using the data from the Bayesian method with 1/αβ prior, and the green trace plots are for 1/α prior. The plots show the stability of the sampled data since they contain the initial values indicated by the horizontal lines and have almost uniform spikes upwards or downwards.

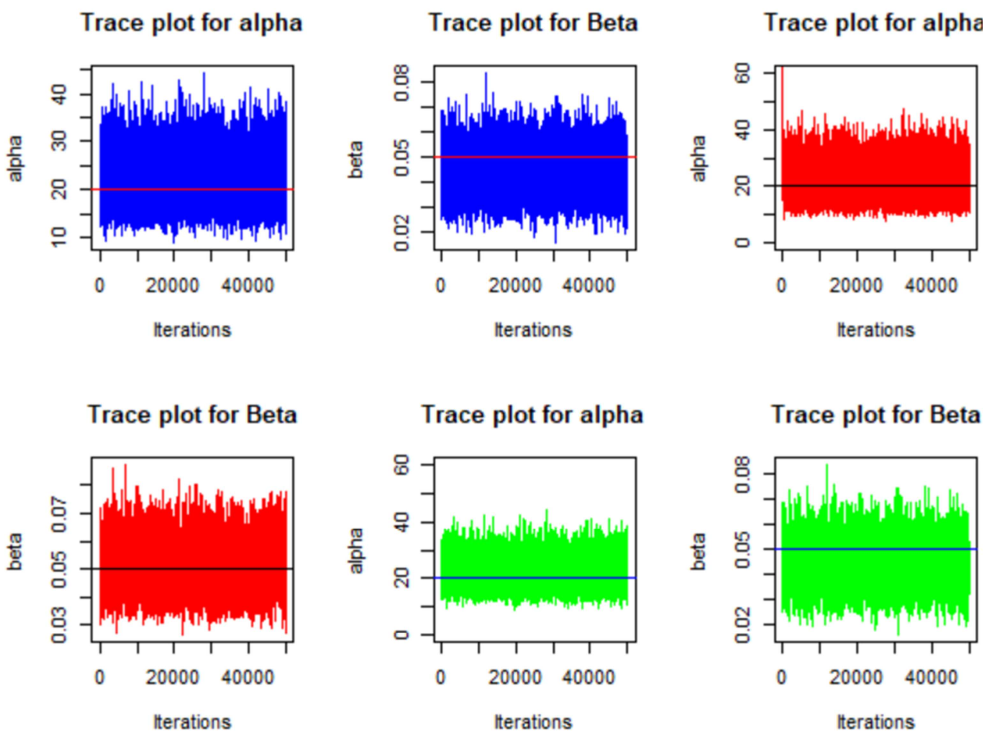


***Figure 1.*** *Trace plots for α and β generated using the three priors.*

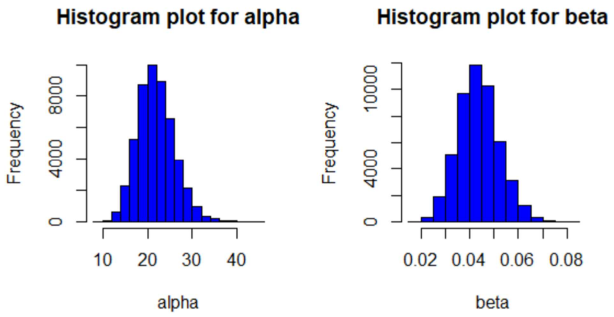


***Figure 2.*** *Histogram for α and β generated using informative prior.*

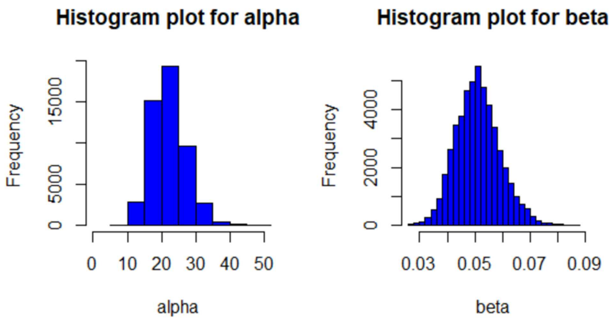


***Figure 3.*** *Histogram for α and β generated using* $1/\alpha\beta$ *prior.*

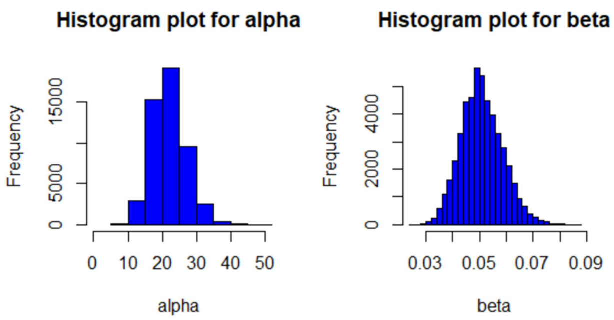


***Figure 4.*** *Histogram for α and β generated using* $1/\alpha$ *prior.*

Another issue of interest was to examine the marginal posterior distribution of the two parameters. Figures 2, 3, and 4 show histograms for $\alpha$ and $\beta$ constructed using the three Bayesian methods (gamma-distributed informative prior and non-informative priors; 1/αβ and 1/α. Since the histograms are approximately normally distributed, we share the value of $\gamma$ equally to both tails of the marginal posterior distribution of each parameter.

A total of 5000 samples of interfailure times data were generated and used to sample 5000 datasets, each for $\alpha$ and $\beta$ from the three posterior distributions. The samples had an average size of 23 interfailure times. An R code was developed such that the same 5000 samples of interfailure times data were used for the four methods (Wald and three Bayesian approaches) for accurate comparisons. For the Wald technique, the datasets were used with the Log-likelihood function to compute parameter estimates, construct confidence intervals, and generate coverage probabilities through repeated sampling. Moreover, Bayesian credible intervals and coverage probabilities were obtained for each sample of $\alpha$ and $\beta$ random variables generated from the posterior distributions. In this regard, each method had a total of 5000 intervals. Interval lengths were obtained for each confidence and credible interval as the difference between the upper and lower bounds. Coverage probabilities for each method were estimated as the proportion of the 5000 confidence and credible intervals, which contain the true parameter value. The coverage probabilities were recorded as obtained, while summary statistics (minimum, maximum, mean, and standard deviation) of widths of the 5000 confidence and credible intervals for each method are presented in Table 1. The results indicate that two Bayesian methods (with informative and *1/αβ* priors) have shorter average interval lengths than the Wald approach. Of the methods, Bayesian with informative prior was superior, yielding higher coverage probabilities and shorter average widths.

***Table 1.*** *Estimated summary statistics and coverage probabilities of widths of the 95% Bayesian credible and Wald confidence intervals for the parameters* $\alpha$ *and* $\beta$ *on the basis of 5000 samples when T = 100,* $\alpha_0 = 20$, *and* $\beta_0 = 0.05$.

| Method | | Min | Max | Mean | Std dev | Cp |
|---|---|---|---|---|---|---|
| Wald | $\alpha$ | 11.40 | 61.40 | 18.83 | 3.7986 | 0.943 |
| | $\beta$ | 0.02811 | 0.09064 | 0.04123 | 0.00685 | 0.95 |
| Bayesian (Informative) | $\alpha$ | 11.48 | 19.12 | 15.06 | 1.2149 | 0.988 |
| | $\beta$ | 0.02297 | 0.07146 | 0.03665 | 0.00610 | 0.960 |
| Bayesian ($1/\alpha\beta$) | $\alpha$ | 11.18 | 25.07 | 17.79 | 2.1118 | 0.95 |
| | $\beta$ | 0.02098 | 0.08831 | 0.03627 | 0.00771 | 0.934 |
| Bayesian ($1/\alpha$) | $\alpha$ | 11.60 | 855.77 | 22.50 | 31.12 | 0.955 |
| | $\beta$ | 0.02805 | 0.09369 | 0.04190 | 0.00711 | 0.934 |

The estimated coverage probabilities of Wald confidence and Bayesian credible intervals for $\alpha$ and $\beta$ were reasonably close to the theoretical value of 0.95, as shown in Table 1. Moreover, trace plots were constructed for the interval lengths of $\alpha$ and $\beta$, as shown in Figures 5 and 6. All the plots were constructed using the same limits on the y-axis for easier comparison. It can be noticed that the interval lengths for the Bayesian credible intervals using the joint gamma-distributed informative prior were shorter than both for Wald and non-informative priors. For the parameter $\alpha$, the interval lengths generated using the gamma-distributed informative and *1/αβ* priors had shorter and uniform downward and upward spikes, as displayed in Figure 5, the second and third graphs. The behavior of the graphs shows that the two interval estimation methods have increased precision, quantified by smaller standard deviation values in Table 1. The first graph in Figure 5, generated using the Wald approach, shows long upward spikes but almost uniform downward spikes. However, the fourth graph has long upward spikes, suggesting that the method would yield larger credible intervals than Wald and the other two Bayesian methods. The large standard deviation of 31.12 quantifies the long upward variable spikes.

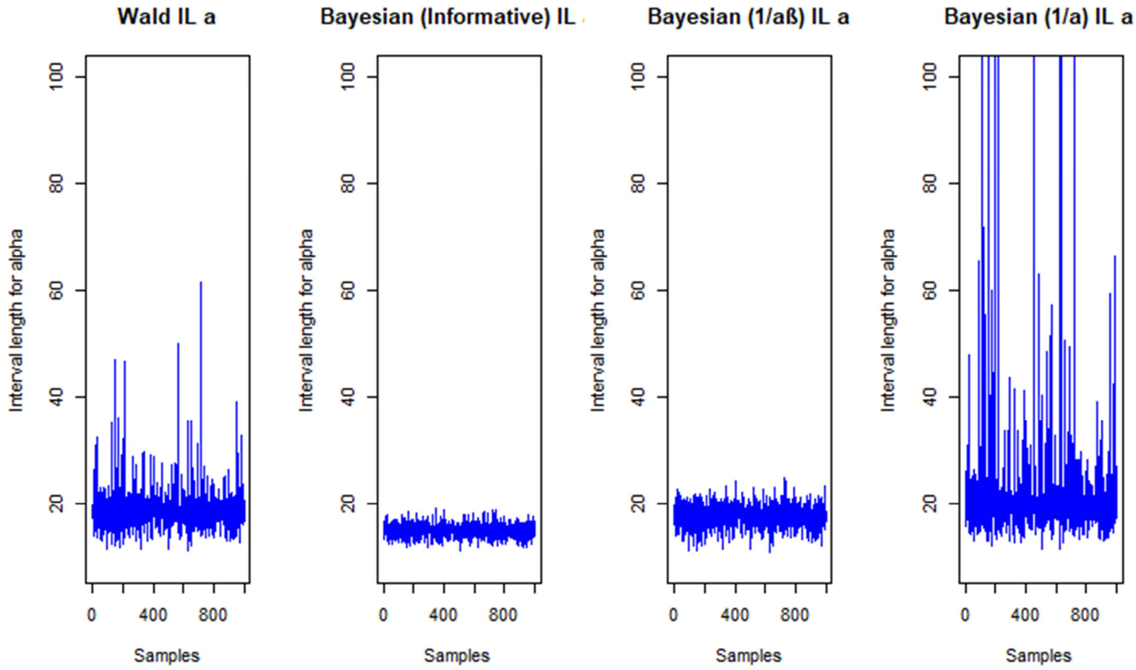


***Figure 5.*** *Trace plots for interval lengths of α.*

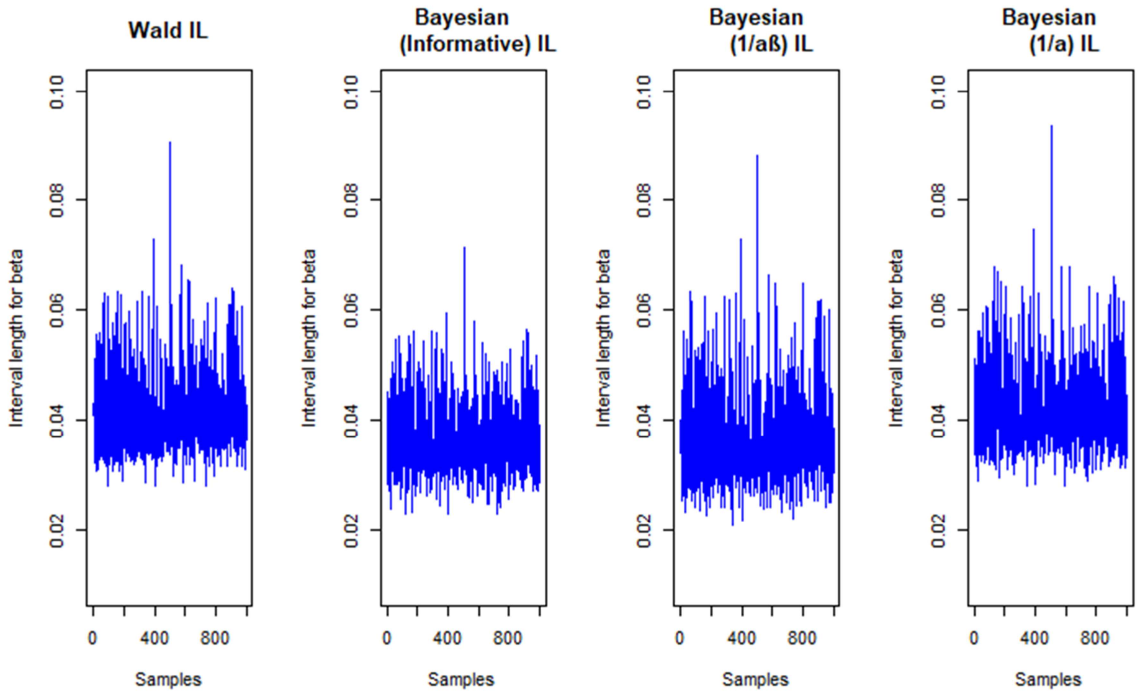


***Figure 6.*** *Trace plots for interval lengths of β.*

In Figure 6, the spikes for the interval lengths for the Wald confidence and Bayesian credible intervals show a similar pattern. The downward spikes for the four trace plots are shorter and tend to be uniform. However, the long upward spikes are fewer and a bit shot in the second graph generated using the Bayesian method with gamma-distributed informative prior. Smaller and slightly different standard deviations for $\beta$ reported in Table 1, column 6, indicate the reason for similar behavior in the graphs.

## 6. Conclusion

This article explored interval estimation in an NHPP with a Delayed S-shaped intensity function using the Wald and Bayesian methods. Parameter estimation is a crucial step in software reliability assessment because the obtained values can be used in prediction. Commonly explored approaches to assess the performance of different interval estimation methods include obtaining interval lengths and coverage probabilities. We used Bayesian approaches with informative and non-informative priors to identify better methods that enhance estimation accuracy in software reliability assessment. We compared the two methods using coverage probabilities and interval lengths on the basis of simulated data. Through repeated sampling, 5000 samples were generated and used to construct confidence and credible intervals and to compute coverage probabilities. The Bayesian method using the gamma-distributed informative prior yielded credible regions with shorter lengths, and higher coverage probabilities compared to the Wald and other Bayesian approaches. Moreover, the Bayesian method using a non-informative prior given by $1/\alpha\beta$ also yielded shorter interval lengths than the Wald approach. However, the $1/\alpha$ non-informative prior yielded unstable results for $\alpha$, indicated by the long upward spikes in the last graph in Figure 5. Thus, the method may not be appropriate because it contains implausible values of the two parameters. The study results indicate that the Bayesian approach is more important, appropriate, and precise in estimating the parameters of the Delayed S-shaped NHPP software reliability model.

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